Name: SOLUTIONS

Score: ___ / 55

3CD MAS Test – May 2010
Calculator Free

1. **[4 marks]**

4m

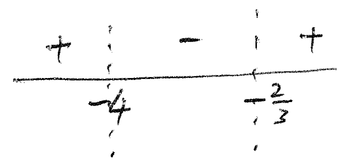
Solve $|2x + 3| > |x - 1|$, showing full reasoning.

$$(2x+3)^2 > (x-1)^2$$

$$4x^2 + 12x + 9 > x^2 - 2x + 1$$

$$3x^2 + 14x + 8 > 0$$

$$(3x+2)(x+4) > 0$$



$$\therefore \quad \underline{\underline{x < -4}} \quad \text{or} \quad \underline{\underline{x > -\tfrac{2}{3}}}$$

✓ Squares both sides.
✓ Factorises LHS.
✓ Determines correct regions.
✓ Final answer, correct notation.

2. **[3 marks]**

3m

Determine the exact value of $\lim_{h \to 0}\left(\dfrac{\cos 2(a+h) - \cos 2a}{h}\right)$, where a is a constant.

$$\lim_{h \to 0}\left(\frac{\cos 2(a+h) - \cos 2a}{h}\right) = \frac{d}{dx}\left[\cos 2x\right]\Big|_{x=a}$$

$$= \left[-2\sin 2x\right]\Big|_{x=a}$$

$$= \underline{\underline{-2\sin 2a}}$$

Recognize $\frac{d}{dx}$ ✓
Identify function: $\cos 2x$ ✓
Derivative evaluated at $x=a$ ✓

Σ 7m

6m **3.** **[6 marks]**

The equation of one of the tangents to the curve $xy(x+y) - 12 = 0$ at the points where $x = 1$ is $y = -\frac{15x}{7} + \frac{36}{7}$.

Determine the equation of the other tangent to the curve when $x = 1$.

$$x^2y + xy^2 - 12 = 0$$

$$2xy + x^2\left(\frac{dy}{dx}\right) + y^2 + 2xy\left(\frac{dy}{dx}\right) = 0$$

For $x = 1$: $y + y^2 - 12 = 0$

$$(y+4)(y-3) = 0$$

$$y = -4 \text{ or } y = 3$$

At $(1, 3)$: $6 + \frac{dy}{dx} + 9 + 6\left(\frac{dy}{dx}\right) = 0$

$$\therefore \frac{dy}{dx} = \frac{-15}{7}$$

At $(1, -4)$: $-8 + \frac{dy}{dx} + 16 - 8\left(\frac{dy}{dx}\right) = 0$

$$\therefore \frac{dy}{dx} = \frac{8}{7}$$

At $(1, -4)$: $(y+4) = \frac{8}{7}(x-1)$

$$\therefore y = \frac{8}{7}x - \frac{8}{7} - 4$$

$$\underline{\underline{y = \frac{8}{7}x - \frac{36}{7}}} \quad \text{or} \quad \boxed{8x - 7y = 36}$$

Determines 2 pts: $(1,3)$ and $(1,-4)$ ✓
Differentiates implicitly ✓✓
Determines gradients ✓
Determines which point to use ✓
Determines equation of tangent. ✓

Σ 6m

4. **[10 marks]**

For each of the following functions, find $\frac{dy}{dx}$.

3m (a) $y = \frac{x^3}{\cos x}$

$$\frac{dy}{dx} = \frac{3x^2(\cos x) - x^3(-\sin x)}{\cos^2 x}$$

$$= \frac{x^2(3\cos x + x\sin x)}{\cos^2 x}$$

Uses Quotient rule. ✓
Correct differentiation. ✓
Final answer expressed succinctly. ✓

[3]

4m (b) $y = (\sin x)^x$

$$\ln y = x \ln(\sin x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{\sin x} \cdot \cos x + \ln(\sin x)$$

$$\therefore \frac{dy}{dx} = \left[\frac{x\cos x}{\sin x} + \ln(\sin x)\right] \cdot (\sin x)^x$$

✓ Takes natural log of both sides.
✓✓ Implicitly differentiates equation.
✓ Expresses answer in terms of x.

[4]

3m (c) $y = \frac{t+2}{t}$ and $x = \frac{t-2}{2}$, giving your answer in terms of x.

$$\frac{dy}{dt} = \frac{-2}{t^2}, \quad \frac{dx}{dt} = \frac{1}{2}$$

$$\therefore \frac{dy}{dx} = \frac{-2}{t^2} \times \frac{2}{1} = \frac{-4}{t^2}$$

$$= \frac{-4}{(2x+2)^2}$$

✓ Correctly finds $\frac{dy}{dt}$ and $\frac{dx}{dt}$.
✓ Calculates $\frac{dy}{dx}$.
✓ Expresses answer in terms of x.

[3]

Σ 10m

5. **[6 marks]**

6m

The volume of a cylinder is constant at 50π cm^3, but both the height and the radius are changing. Determine the rate at which the radius is changing at the instant when the height is decreasing at a rate of 3 cm/sec and the radius is 5 cm.

$$\pi r^2 h = 50\pi$$

$$r^2 h = 50$$

$$h = \frac{50}{r^2}$$

$$\therefore \frac{dh}{dr} = \frac{-100}{r^3}$$

$$\left.\frac{dh}{dr}\right|_{r=5} = \frac{-100}{125} = \frac{-4}{5}$$

$$\frac{dr}{dt} \times \frac{dh}{dr} = \frac{dh}{dt}$$

$$\left(\frac{dr}{dt}\right)\left(-\frac{4}{5}\right) = -3$$

$$\therefore \frac{dr}{dt} = -3 \times \left(-\frac{5}{4}\right)$$

$$= \frac{15}{4}$$

$$= 3.75 \text{ cm/sec}$$

✓ Expresses Volume = 50π, in terms of r, h.

✓ Differentiates to find $\frac{dh}{dr}$.

✓ Evaluates $\frac{dh}{dr}$ at $r = 5$.

✓ Uses correct relationship, ie, $\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dt}$

✓ Substitutes correct information, and $\frac{dh}{dt} < 0$

✓ Determines $\frac{dr}{dt}$ correctly.

Σ 6m

6. **[9 marks]**

Consider the function $y = |1 - 2x| + |x|$.

(a) Rewrite the function in piecewise form.

3m

$$y = \begin{cases} 1 - 3x, & x < 0 \\ 1 - x, & 0 \le x \le \frac{1}{2} \\ 3x - 1, & x > \frac{1}{2} \end{cases}$$

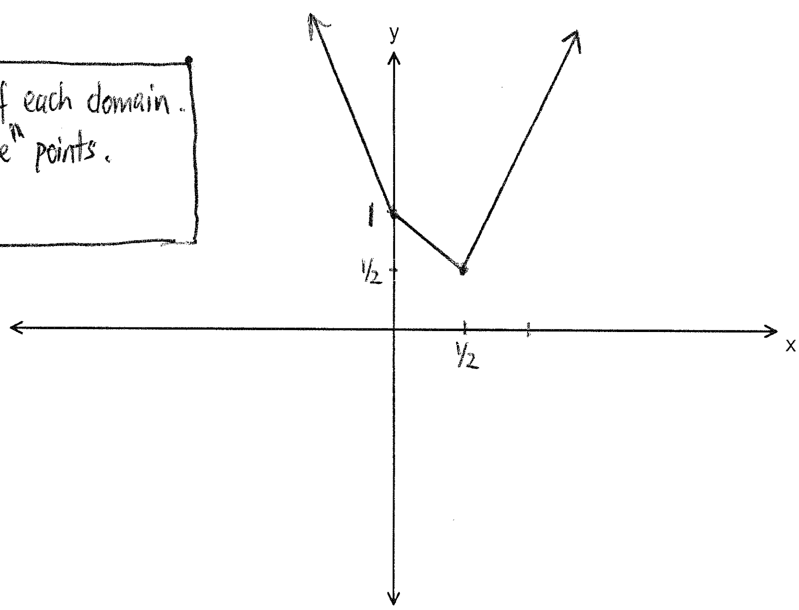
✓ Identifies critical points.
✓ Determines correct piecewise functions
✓ Uses correct notation.

[3]

3m

(b) Sketch a graph of the function on the axes provided.

✓ Correctly draws a sketch of each domain.
✓ Graph displays correct "fracture" points.
✓ Graph is drawn accurately.



[3]

(c) Hence, differentiate the function with respect to x.

3m

$$\frac{dy}{dx} = \begin{cases} -3, & x < 0 \\ -1, & 0 < x < \frac{1}{2} \\ 3, & x > \frac{1}{2} \end{cases}$$

✓ Differentiates each component.
✓ Excludes $x = 0$, $x = \frac{1}{2}$ from domain
✓ Uses appropriate notation.

[3]

Σ 9m

4m 7. **[4 marks]**

Determine the following limit, showing full reasoning.

$$\lim_{x\to 0}\left(\frac{\tan^2 x}{1-\cos x}\right) = \lim_{x\to 0} \frac{\sin^2 x}{\cos^2 x(1-\cos x)}$$

$$= \lim_{x\to 0} \frac{(1+\cos x)(1-\cos x)}{\cos^2 x\ (1-\cos x)}$$

$$= \lim_{x\to 0} \frac{1+\cos x}{\cos^2 x}$$

$$= \frac{1+1}{1^2}$$

$$= \underline{\underline{2}}$$

✓ Expresses $\tan^2 x$ as $\frac{\sin^2 x}{\cos^2 x}$

✓ Factorises and removes Common Factor.

✓ Uses $\lim_{x\to 0} \cos x = 1$.

✓ Evaluates correctly.

3m 8. **[3 marks]**

Explain clearly how you would determine the following derivative.
(It is not necessary to work out the answer.)

$$\frac{d}{d\left(\sqrt{x}\right)}\ln\left[\frac{2\sqrt{x}}{1-\sqrt{x}}\right]$$

Rewrite as $\frac{d}{du}\left[\ln\left(\frac{2u}{1-u}\right)\right]$.

Use logarithm laws to simplify logarithm.

Differentiate using chain rule.

Substitute u for $\sqrt{x}$ into final answer.

✓ Rewrites.
✓ Simplifies log.
✓ Differentiates and substitutes.

Σ 7m

9. **[10 marks]**

Given matrices $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} x & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 8 & -3 \\ 5 & 1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 2 \\ y \end{bmatrix}$,

(a) Determine $\mathbf{A} + \mathbf{D}$.

2m

Undefined, since A and D have different dimensions.

States answer ✓
Gives reason ✓

[2]

(b) If $\mathbf{AB} = \mathbf{C}$, then determine the value of x.

4m

$$\begin{bmatrix} 4x & -3 \\ 2x & 1 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow$ $4x = 8$ and $2x = 5$

$x = 2$ and $x = 2\frac{1}{2}$

since solution for x is not unique,

$\therefore$ There is no solution for x

✓ Correctly evaluates AB.
✓ Determines 2 solutions for x.
✓ Explains x is not unique.
✓ Final answer: no sol^{n}.

[4]

(c) If $\mathbf{A} + 2\mathbf{B} = \begin{bmatrix} 8 & 3 \\ 2 & -3 \end{bmatrix}$, then determine the value of x.

2m

$$\begin{bmatrix} 4+2x & 3 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 2 & -3 \end{bmatrix}$$

$\Rightarrow 4 + 2x = 8$

$x = 2$

✓ Correctly evaluates A+2B.
✓ Solves for x

[2]

(d) If $x = 2\sqrt{2}$, then determine $\mathbf{B}^2$.

2m

$$B^2 = \begin{bmatrix} x & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}$$

✓ Correctly evaluates B^2
✓ Substitutes to give final answer.

[2]

END OF TEST

Σ 10m